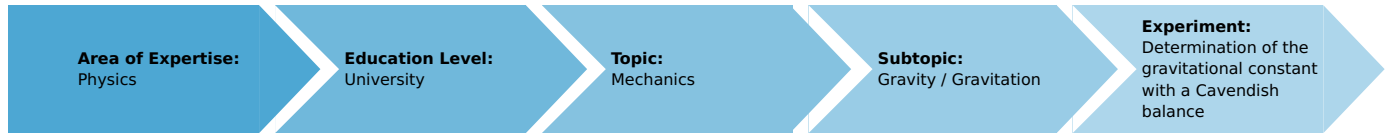


Determination of the gravitational constant with a Cavendish balance (Item No.: P2130901)

Curricular Relevance



Difficulty



Difficult

Preparation Time



20 Minutes

Execution Time



20 Minutes

Recommended Group Size



2 Students

Additional Requirements:

Experiment Variations:

Keywords:

Law of gravitation, free and damped oscillations, torsional vibrations, forced oscillations

Introduction

Overview

Two small lead spheres of equal mass are positioned one at each end of a beam which is held suspended by a thin tungsten thread, so that it can swing freely across its equilibrium position. When two further, but larger, lead spheres held on a swivel arm are now brought near to the small lead spheres, forces of attraction resulting from gravitation effect acceleration of the small spheres in the direction of the larger spheres. At the same time, the twisted metal thread generates a restoring moment of rotation, so that the beam is subjected to damped oscillation across a new equilibrium position. The gravitational constant can be determined both from the difference in the angle of rotation of the different equilibrium positions and from the dynamic behaviour of the swinging system during attraction. An integrated capacitive sensor produces a direct voltage that is proportional to the angle of deflection. This can be recorded over time by an interface system, and the value of the angle of rotation that is required be so determined.

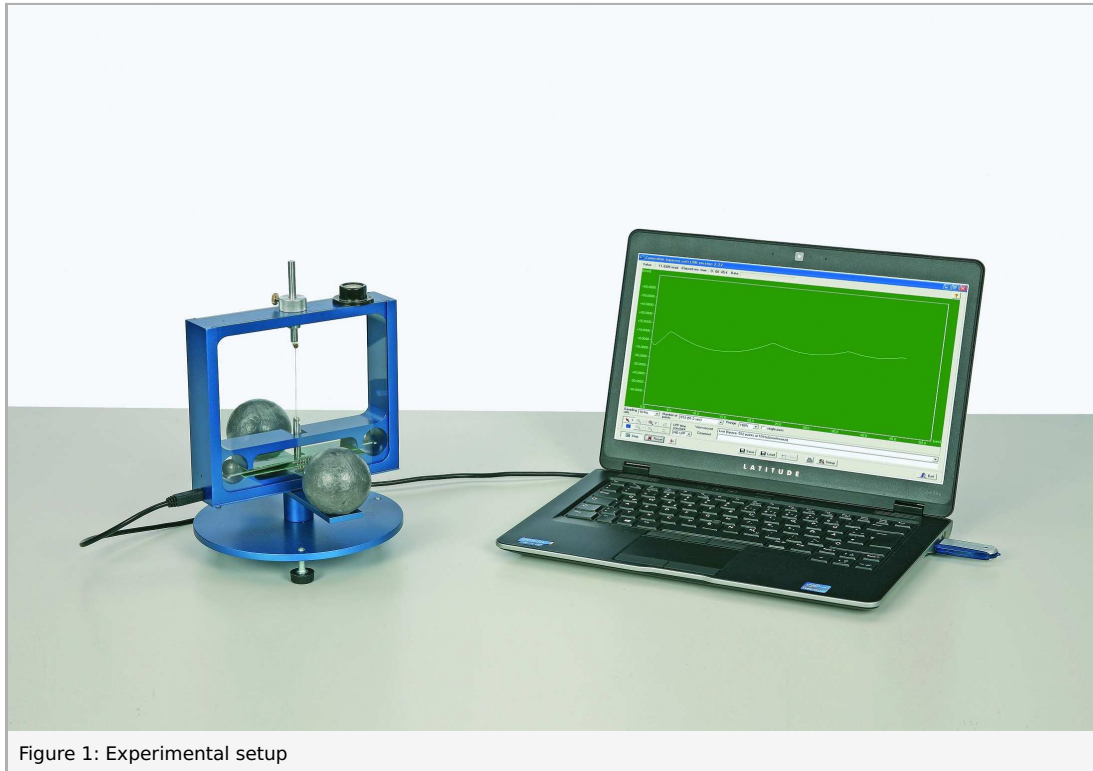


Figure 1: Experimental setup

Equipment

Position No.	Material	Order No.	Quantity
1	Cavendish balance/computerized	02540-00	1
2	Circular level, d = 36 mm	02123-00	1
Recommended:			
	Optical base plate in covering case	08700-01	1

Tasks

1. If necessary, calibrate the capacitive angle sensor.
2. Determine the gravitational constant, using either the acceleration method, the final deflection method or the resonance method. Determine the gravitational constant, using either the equilibrium method or the resonance method.
3. Determine the time of oscillation and the damping of the freely swinging torsion pendulum.

Set-up and procedure

General advice concerning the use of the Cavendish balance

When using the Cavendish balance, it is important to bear in mind a few simple rules concerning the handling of the balance:

1. The Cavendish balance is a high-precision instrument that needs to be handled and operated with extreme care.
2. As the Cavendish balance is highly sensitive to mechanical vibrations, you may wish to place the balance on any kind of support plate that will enhance the damping of the system, e.g. all example measurements in this document were carried out on a PHYWE Optical base plate in covering (08700-01). If possible, do not position the balance and your computer on the same table, thus preventing the coupling of vibrations from your computer table into the balance. Make sure that the USB cable that connects the balance to your computer is not under mechanical stress. Avoid touching the balance and the table if not necessary in order to prevent mechanical vibrations coupling into the system, and carefully close the door when leaving the room as a compressional air wave can heavily affect the system.
3. The Cavendish balance is not screened against electrostatic forces. As the small lead spheres may be electrostatically charged, it is necessary to remove any electrostatic influence from the vicinity of the balance.
4. Make sure to avoid large temperature fluctuations in the lab during the experiment.
5. It is strongly recommended to set up and prepare the Cavendish balance the day before performing the experiment, allowing the system to reach near-equilibrium conditions before starting the measurement.
6. When performing the experiment, please be aware that the gravitational force of your own body has significant influence on the measurement. For this reason, do not approach the Cavendish balance if not necessary, and leave the lab during the measurement.

When choosing a location for the balance in your lab, make sure to minimize any mechanical or electrostatic influence on the balance!

Software Installation and Sensor calibration

Install the „Cavendish balance unit USB“ software on your computer („Cavendish USB v2.23.exe“). Run the „CDM_Setup.exe“ for driver installation. You may check for the most recent driver on the manufacturer's website. Connect the balance to your computer using the USB cable, and start the software.

The position of the beam is detected by a capacitive sensor which is pre-calibrated by the manufacturer. If there is need to re-calibrate the sensor, perform the following steps:

1. Rotate the tungsten thread by carefully turning the screw on top of the balance until one edge of the beam touches the glass window.
2. Use the „Setup“ button to open the setup dialogue. Clicking on one of the „Adjust“ symbol will set this position to a value of ± 75 mrad.
3. Rotate the tungsten thread in the opposite direction by turning the screw on top of the balance until the edge of the beam touches the opposite glass window.
4. Clicking on the other „Adjust“ symbol will set this position as the second boundary.
5. Confirm the dialogue using the „Ok“ button.

If the edge of the beam sticks to the glass, loosen it by gently tapping the top of the balance.

It is strongly recommended to perform the adjustment procedure the day before performing the experiment, allowing the system to reach near-equilibrium conditions.

Please note that even when using a vibration-damping support and allowing the system to equilibrate, this interference signal will still be in the order of magnitude of at least 10% of the measured values in the actual experiment. Fig. 2 shows an 18 hours measurement, exhibiting oscillations both in the time scale of a few minutes and of several hours, and a linear drift, without any deliberate external manipulation of the system. Although the fluctuation on the larger time scale will certainly exceed the duration of the experiment, the oscillation shown in Fig. 3 (detail of Fig. 2) will considerably contribute to the measurement error.

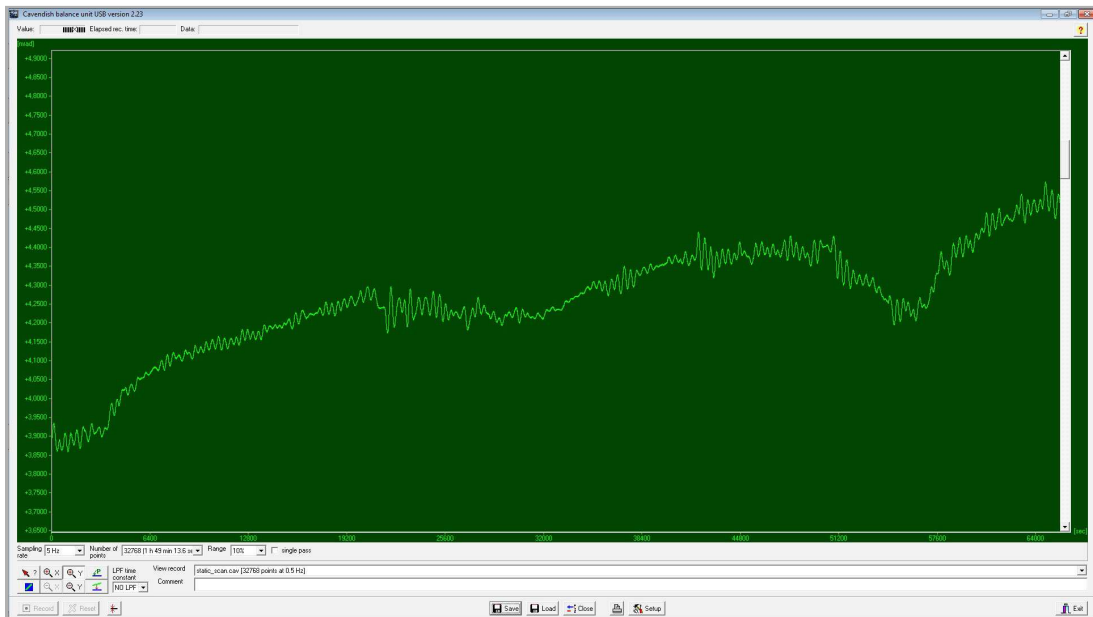


Figure 2: 18 hours measurement without external manipulation of the system.

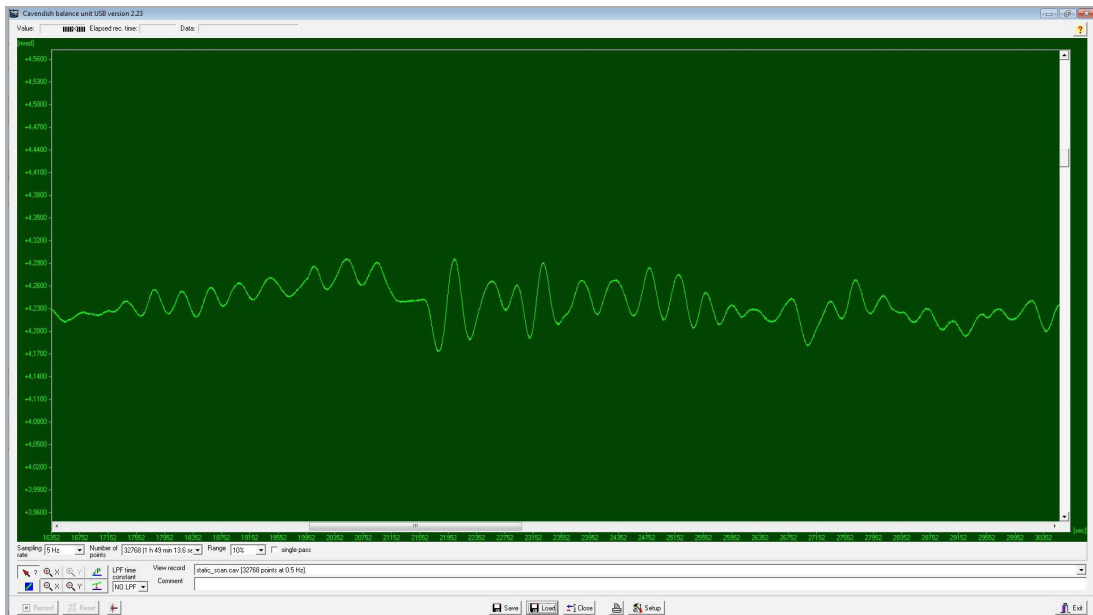


Figure 3: Detail of Fig. 2.

Equilibrium method

Before performing the experiment, you may wish to use Newton's Law of Gravitation to calculate the gravitational force of your own body on the small lead spheres for different distances to the setup, and compare these values to the gravitational force of the large lead spheres on the small lead spheres. When considering equivalent forces in both situations, knowledge of the gravitational constant G (or the mass of the small lead sphere, m) is not mandatory for this calculation.

Example: Consider the mass of an average human body, $M_{\text{body}} = 75 \text{ kg}$ in a distance r_{body} to the small lead sphere, and the mass of a large lead sphere $M = 1 \text{ kg}$, which has its center of mass in a distance $R = 5 \text{ cm}$ to the center of mass of the small lead sphere of mass m . If the gravitational force F of both your body and the large lead sphere on the small lead sphere are equal, this yields

$$F = G \frac{m \cdot M}{R^2} = G \frac{m \cdot M_{\text{body}}}{r_{\text{body}}^2} \rightarrow \frac{M}{R^2} = \frac{M_{\text{body}}}{r_{\text{body}}^2} \quad (1)$$

$$r_{\text{body}}^2 = \frac{M_{\text{body}}}{M} \cdot R^2 \rightarrow r_{\text{body}} = \sqrt{\frac{M_{\text{body}}}{M}} \cdot R = \sqrt{\frac{75 \text{ kg}}{1 \text{ kg}}} \cdot 5 \text{ cm} \approx 43 \text{ cm} \quad (2)$$

In other words, positioning your own body in the distance of 0.43 m to the balance has the same effect on the small lead spheres as the large lead spheres do during the experiment! This is why it is mandatory to stay away from the setup whenever it is possible.

Perform the experiment according to the following steps:

1. Start „Cavendish balance unit USB“ software.
2. Choose „Sampling rate“ and „Number of points“, e.g. 5 Hz and 32768 points.
3. Decrease the „Range“ value. The amplitude will be in the order of magnitude of a few mrad, and a range of 5% or 10% will be sufficient.
4. Make sure that the Low-pass-filter („LPF“) is set to „No LPF“.
5. You may wish to use the „Adjust zero“ button (next to the „Record“ und „Reset“ buttons), e.g. if the system already reached near-equilibrium condition (see Fig. 4). This step is not mandatory since it is possible to derive the value for the gravitational constant solely based on the difference between the two respective rest positions.
6. Start the measurement („Record“ button).
7. Rotate the swivel arm that holds the large lead spheres until the spheres gently touch the glass. Leave the vicinity of the setup and record at least three oscillations.
8. Rotate the swivel arm in the opposite direction, again until the spheres gently touch the glass. Leave the vicinity of the setup and record at least three oscillations.
9. Repeat these alternating rotations as many times as you wish.
10. Stop the measurement („Stop“ button) and save your data („Save“ button).

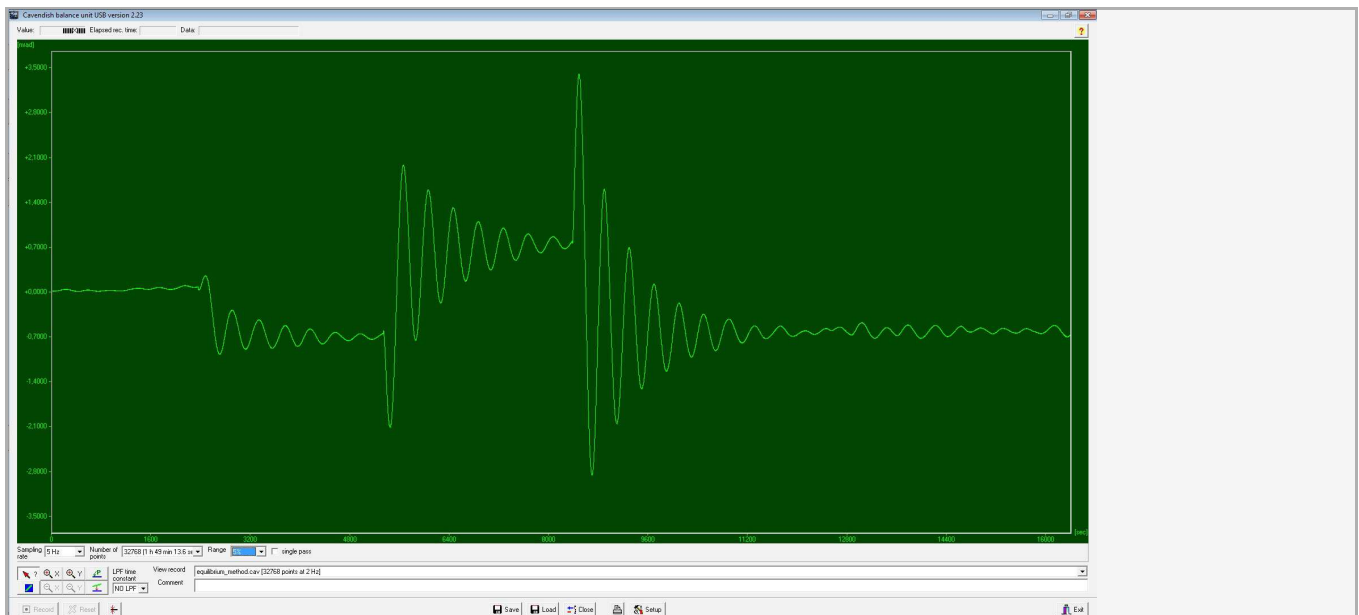


Figure 4: Equilibrium method. Note the change in amplitude during the different steps of the measurement, and the interference signal at the end of the measurement

Resonance method

Perform the experiment according to the following steps:

1. Start „Cavendish balance unit USB“ software.
2. Choose „Sampling rate“ and „Number of points“, e.g. 10 Hz and 12481 points.
3. Decrease the „Range“ value. The amplitude will be in the order of magnitude of a 10 mrad peak-to-peak, and a range of 10% will be sufficient.
4. Make sure that the Low-pass-filter („LPF“) is set to „No LPF“.
5. You may wish to use the „Adjust zero“ button (next to the „Record“ und „Reset“ buttons), e.g. if the system already reached near-equilibrium condition. This step is not mandatory.
6. Start the measurement („Record“ button).
7. Rotate the swivel arm that holds the large lead spheres until the spheres gently touch the glass. Leave the vicinity of the setup and observe.
8. When the value reaches its turning point rotate the swivel arm in the opposite direction, again until the spheres gently touch the glass.
9. Repeat the rotation at each turning point until the maximum amplitude reaches a constant value.
10. Stop the measurement („Stop“ button) and save your data („Save“ button).

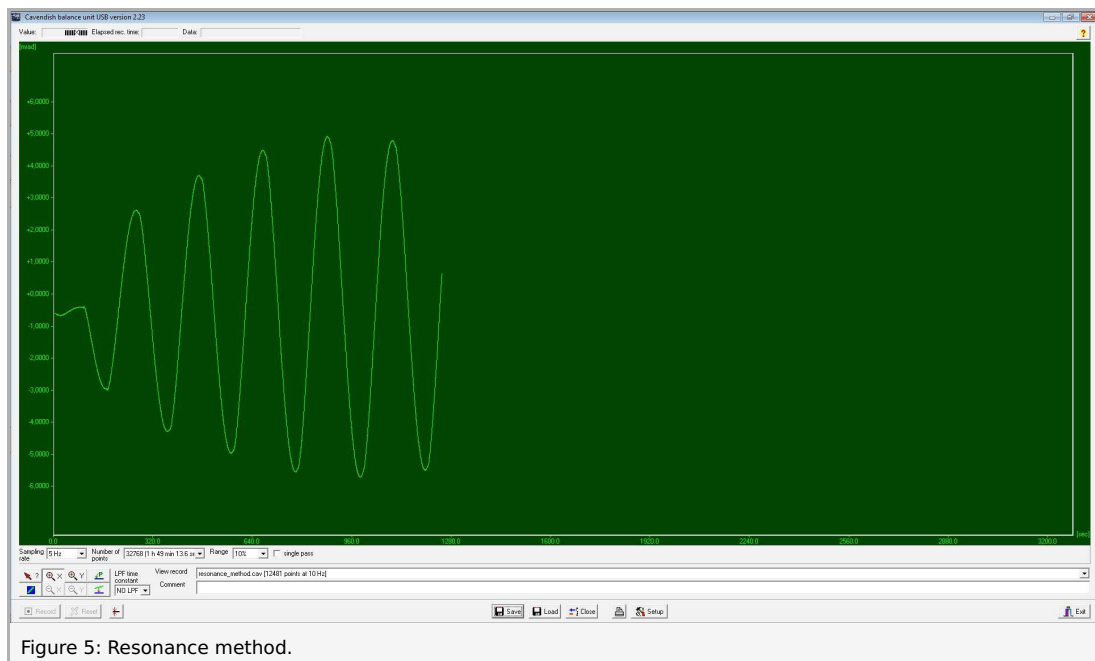


Figure 5: Resonance method.

Damped oscillation

Perform the experiment according to the following steps:

1. Start „Cavendish balance unit USB“ software.
2. Choose „Sampling rate“ and „Number of points“, e.g. 10 Hz and 32768 points.
3. Decrease the „Range“ value. The amplitude will be in the order of magnitude of a 3 mrad peak-to-peak, and a range of 5% will be sufficient.
4. Make sure that the Low-pass-filter („LPF“) is set to „No LPF“.
5. Start the measurement („Record“ button).
6. Rotate the swivel arm that holds the large lead spheres until the spheres are positioned in between their extremal positions. Leave the vicinity of the setup and observe several damped oscillations.
7. Stop the measurement („Stop“ button) and save your data („Save“ button).

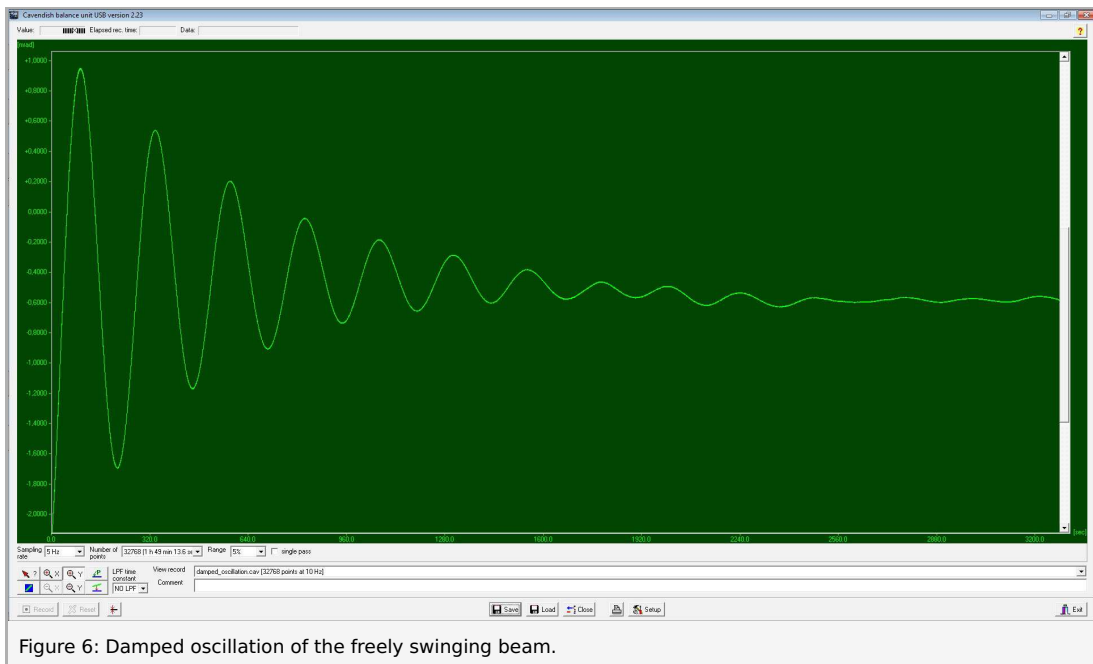


Figure 6: Damped oscillation of the freely swinging beam.

Theory and evaluation

Determination of the damping constant λ and oscillation period T

You will need to find the oscillation period T of the freely swinging beam, as this quantity will be used in order to determine the gravitational constant from the equilibrium or resonance method.

Equation (3) describes the change in angle over time $\alpha(t)$ for the freely swinging beam.

$$\alpha(t) = \alpha_0 + \alpha^* \cdot e^{-\lambda t} \cos(\omega t) \quad (3)$$

(α_0 = the voltage of the equilibrium position, generally $\alpha_0 \neq 0$;

α^* = initial amplitude; λ = damping constant)

Using

$$x = e^{-\frac{1}{2}\lambda T} \quad (4)$$

we obtain from equation (3):

$$t_1 = 0 \rightarrow \alpha_1 = \alpha_0 + \alpha^* \quad (5.1)$$

$$t_2 = \frac{T}{2} \rightarrow \alpha_2 = \alpha_0 - (\alpha_1 - \alpha_0) \cdot x \quad (5.2)$$

$$t_3 = T \rightarrow \alpha_3 = \alpha_0 + (\alpha_1 - \alpha_0) \cdot x^2 \quad (5.3)$$

$$t_n = (n-1)\frac{T}{2} \rightarrow \alpha_n = \alpha_0 + (\alpha_1 - \alpha_0) \cdot x^{n-1} \cdot (-1)^{n-1} \quad (5.4)$$

Using equation (4) for the three successive reversal points α_n , α_{n+1} and α_{n+2} we can find from them the zero line α_0 of the damped oscillation.

$$\alpha_0 = \frac{\alpha_{n+1} + x \cdot \alpha_n}{1 + x} = \frac{\alpha_{n+2} + x \cdot \alpha_{n+1}}{1 + x} \quad (6)$$

Using equation (6) for three successive reversal points we can therefore determine both the dimension $x = \exp(-\frac{1}{2} \cdot \lambda T)$ and also, with x known, the zero line α_0 of the damped oscillation.

$$x = e^{-\frac{1}{2}\lambda T} = \frac{\alpha_{n+2} - \alpha_{n+1}}{\alpha_n - \alpha_{n+1}} \quad (7)$$

A satisfactorily accurate average value for x can be obtained from several triplets of reversal points, whereby a start can be made both from the upper reversal points and from the lower reversal points.

An average value of $x = (0.845 \pm 0.035)$ is obtained for the free and damped oscillations in Fig. 6. Using equation (4) and with $T = 239.3$ s, a damping constant of $\lambda = 1.4 \cdot 10^{-3} \text{ s}^{-1}$ is obtained.

Determination of the gravitational constant using the equilibrium method

When using this method, the gravitational constant is determined from the difference in the rest positions of the pendulum at the two extreme positions of the large lead spheres.

With the large lead spheres in their neutral position and the pendulum at rest, the equilibrium position is given by the angle α_0 . When the large lead spheres are brought to their other extreme position then, because of the attractive force, the pendulum swings across a new rest position α_{01} . When, following this, the large lead spheres are brought to the other extreme position, the pendulum swings across the rest position α_{02} . The angle of deflection α_D required for the calculation of the gravitational constant G is so given by $\alpha_D = 1/2(\alpha_{02} - \alpha_{01})$.

Should it not be possible to reach stable rest positions because of external influences (temperature fluctuations, vibrations), these can be determined from the observation of at least three reversal points of the oscillating torsion pendulum.

If the moment τ_G of the gravitation force F_G is equal to the torsional moment τ_T of the filament, the condition for the rest position of the pendulum is fulfilled:

$$\tau_G = F_G \cdot d = \tau_T = D \cdot \alpha_D = \frac{4\pi^2}{T^2} I \cdot \alpha_D \quad (8)$$

(D = directive moment; I = moment of inertia of the oscillating system)

Inserting the moments of inertia of the balance beam I_b and the small lead spheres $2 \cdot I_s$ in (8), then it follows that (the beam has two holes where the small spheres are placed):

$$I = \frac{1}{12} m_b (l_b^2 + w_b^2) + 2m \left(\frac{2}{5} r_s^2 + d^2 \right) - 2m_h \left(\frac{1}{2} r_h^2 + d^2 \right) = 1.404 \cdot 10^{-4} \text{ kg m}^2 \quad (9)$$

The mass of the balance beam is calculated from the dimensions of the beam and the density of the beam material. The gravitation force F_G has three components:

F_1 = attraction force between the large spheres and the close-by small spheres;

F_2 = attraction force between the large spheres and the father small spheres;

F_3 = attraction force between the large spheres and the beam with two holes.

For the moment of gravitational force τ_G we receive:

$$\tau_G = 2(F_1 \cdot d + F_2 \cdot (-d) + F_3 \cdot d) = \frac{2G \cdot M}{R^2} [(m - m_h)(1 - f_1) + m_b f_2] \cdot d \quad (10)$$

(See Appendix for a detailed calculation of the correction factors f_1 and f_2).

From (8) and (9), we finally find for the gravitational constant G :

$$G = \left(\frac{2\pi}{T} \right)^2 \cdot R^2 \cdot I \cdot \frac{1}{2M \cdot d [(m - m_h)(1 - f_1) + m_b f_2]} \cdot \alpha_D \quad (11)$$

This expression can be used to evaluate the measurement performed by the equilibrium method in the most accurate way, as it accounts for systematic errors by correction factors f_1 and f_2 .

$$G = \left(\frac{2\pi}{T} \right)^2 \cdot R^2 \cdot I \cdot \frac{1}{2M \cdot d [(m - m_h)(1 - f_1) + m_b f_2]} \cdot \alpha_D = 9.726 \cdot 10^{-8} \cdot \alpha_D \frac{\text{m}^3}{\text{kg} \cdot \text{s}^2} \quad (12)$$

To determine the angle α_D the easiest possible way, use both rest positions α_{02} and α_{01} , as described above (see Fig. 4).

You will find example values for all the parameters in Eq. (12) at the end of this document.

Determination of the gravitational constant using the resonance method

When the large lead spheres are brought to their extreme positions in-phase then the oscillation of the balance beam reaches the resonance condition. In the extreme position the distance between the small and large spheres is constant ($\bar{R} = \text{const}$), gravitation force and its moment are also constant. Is α_e the stable rest position of the beam and α_D the amplitude if the large sphere is brought to its extreme position, then $(\alpha_e - \alpha_D)$ is the new rest position. For the time dependent amplitude we receive:

$$\alpha(t) = (\alpha_e - \alpha_D) + (\alpha_1 - (\alpha_e - \alpha_D)) \cdot e^{-\lambda t} \cos(\omega t) \quad (13)$$

(α_1 is the first reversion point; $t = 0$)

For the second reversion point α_2 , if the large sphere is brought to its other extreme position we receive:

$$\alpha_2 = (\alpha_e - \alpha_D) - (\alpha_1 - (\alpha_e - \alpha_D)) \cdot x \quad \text{with } x = e^{-\lambda t} \quad (14)$$

For the third reversion point α_3 ($t = T$), if the large sphere is brought to its initial extreme position we receive:

$$\alpha_3 = (\alpha_e + \alpha_D) - (\alpha_2 - (\alpha_e + \alpha_D)) \cdot x \quad (15)$$

From Eq. (14) and (15):

$$\alpha_D = \frac{[(\alpha_{n+1} - \alpha_e) + (\alpha_n - \alpha_e) \cdot x] \cdot (-1)^n}{1 + x} = \frac{[(\alpha_{n+2} - \alpha_e) + (\alpha_{n+1} - \alpha_e) \cdot x] \cdot (-1)^n}{1 + x} \quad (16)$$

According to Eq. (14) α_e can be calculated from three reversion points and for known x -value α_D can be calculated:

$$\alpha_D = \frac{[x \cdot \alpha_n + (1 - x) \cdot \alpha_{n+1} - \alpha_{n+2}] \cdot (-1)^n}{2(1 + x)} \quad (17)$$

Fig. 5 shows the angle as a function of time for the resonance method. At the point with maximal amplitude the large spheres were brought to their other extreme positions. For the angle of deflection α_D we receive $\alpha_D = 67.46 \cdot 10^{-5} \text{ rad}$.

With this value of α_D and according to Eq. (12) the example measurement yields a gravitational constant of

$$G = 6.56 \cdot 10^{-11} \frac{\text{m}^3}{\text{kg s}^2} \quad (18)$$

(literature value: $G = 6.672 \cdot 10^{-11} \frac{\text{m}^3}{\text{kg s}^2}$)

The resonance method is rather quick than exact, only three reversion points are needed for calculation of the gravitation constant.

Appendix. Consideration of systematic errors

Calculation of f_1

Influence exerted by the large lead spheres on the most distant small lead spheres (see Fig. 7)

$$\tau_2 = 2F_{2y} \cdot (-d) = -2G \frac{M \cdot m}{R^2 + 4d^2} \cdot d \cdot \sin \alpha \quad (19.1)$$

$$= -\frac{2G \cdot M \cdot m}{R^2} \cdot \frac{R \cdot R^2}{\sqrt{(R^2 + 4d^2)^3}} \cdot d \quad (19.2)$$

$$= -\frac{2G \cdot M \cdot m}{R^2} \cdot d \cdot f_1 \quad (19.3)$$

with

$$f_1 = \frac{R^3}{\sqrt{(R^2 + 4d^2)^3}} = 0.0352$$

The moment τ_h of the gravitation force between the large spheres and the point like masses m_h of the beam holes:

$$\tau_h = \frac{2G \cdot M \cdot m_h}{R^2} \cdot d \quad (20)$$

The moment τ_2 of the gravitation force between the large spheres and the most distant small spheres reduced by τ_h :

$$\tau_2 = \frac{2G \cdot M \cdot d}{R^2} \cdot (m \cdot f_1 - m_h) \quad (21)$$

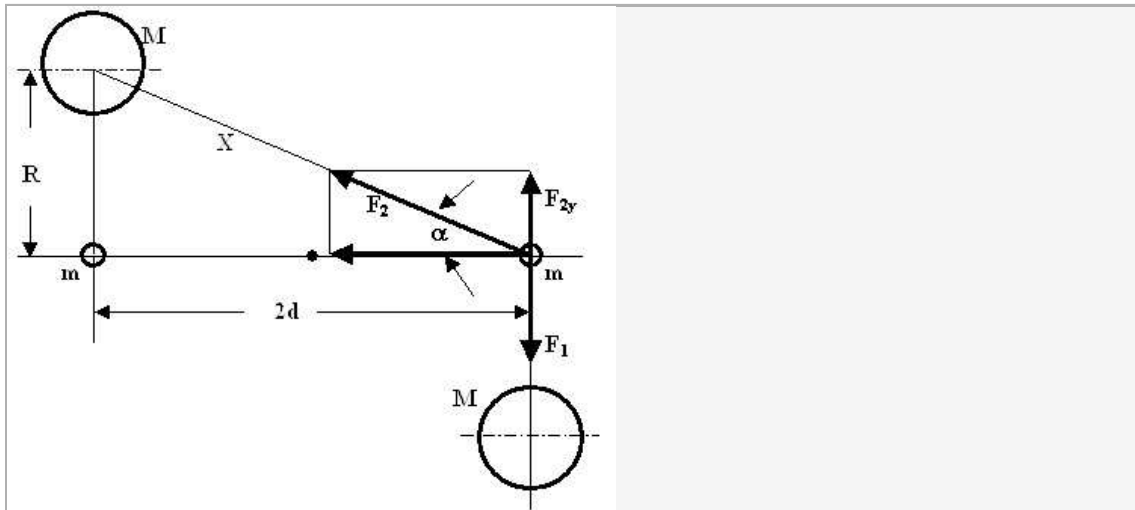


Figure 7: The geometry for calculation of the influence of the large lead spheres on the most distant small lead spheres

Calculation of f_2

Influence exerted by the large lead spheres on the balance beam (see Fig. 8). As the large lead spheres also exert attraction on the balance beam, this influence must also be taken into consideration: When $dm = Q \cdot \rho \cdot dx$ for a mass element of the beam at a distance x from the turning point (Fig. 8).

(Q = cross-sectional area of the beam; ρ = density of the beam material ($Al = 2.7 \text{ g cm}^{-3}$)). A large lead sphere acts on the mass element with a torsional moment:

$$d\tau_3 = F_{3y} \cdot dx = G \cdot \frac{M \cdot Q \cdot \rho}{\sqrt{\left(1 + \left(\frac{d-x}{R}\right)^2\right)^3}} \cdot x dx \quad (22)$$

$$\tau_3 = \frac{G \cdot M \cdot m_b}{l_b} \int_{-\frac{1}{2}l_b}^{+\frac{1}{2}l_b} \frac{x \cdot dx}{\sqrt{\left[1 + \left(\frac{d-x}{R}\right)^2\right]^3}} \quad (23.1)$$

$$= R^3 \int_{-\frac{1}{2}l_b}^{+\frac{1}{2}l_b} \frac{x \cdot dx}{X\sqrt{X}} \quad (23.2)$$

$$= R^3 \left| \frac{-2(bx + 2c)}{\Delta\sqrt{X}} \right|_{-\frac{1}{2}l_b}^{+\frac{1}{2}l_b} \quad (23.3)$$

with $X = ax^2 + bx + c$; $\Delta = 4ac - b^2$; $a = 1$; $b = -2d$; $c = R^2 + d^2$

Taking both large lead spheres into consideration we obtain for τ_3 :

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$$\tau_3 = \frac{2G \cdot M \cdot m_b \cdot d}{R^2} \cdot \frac{R^3}{d \cdot l_b} \left| \frac{1}{\sqrt{1 + \left(\frac{x-d}{R}\right)^2}} \cdot \left(\frac{d(x-d)}{R^2} - 1\right) \right|_{-\frac{1}{2}b}^{+\frac{1}{2}b} \quad (24.1)$$

$$= \frac{2G \cdot M \cdot m_b \cdot d}{R^2} \cdot f_2 \quad \text{with } f_2 = 0.202 \quad (24.2)$$

For the whole moment τ_G we receive (taking both corrections into consideration):

$$\tau_G = \frac{2G \cdot M \cdot d}{R^2} \cdot [(m - m_h) \cdot (1 - f_1 + m_b) \cdot f_2] \quad (25)$$

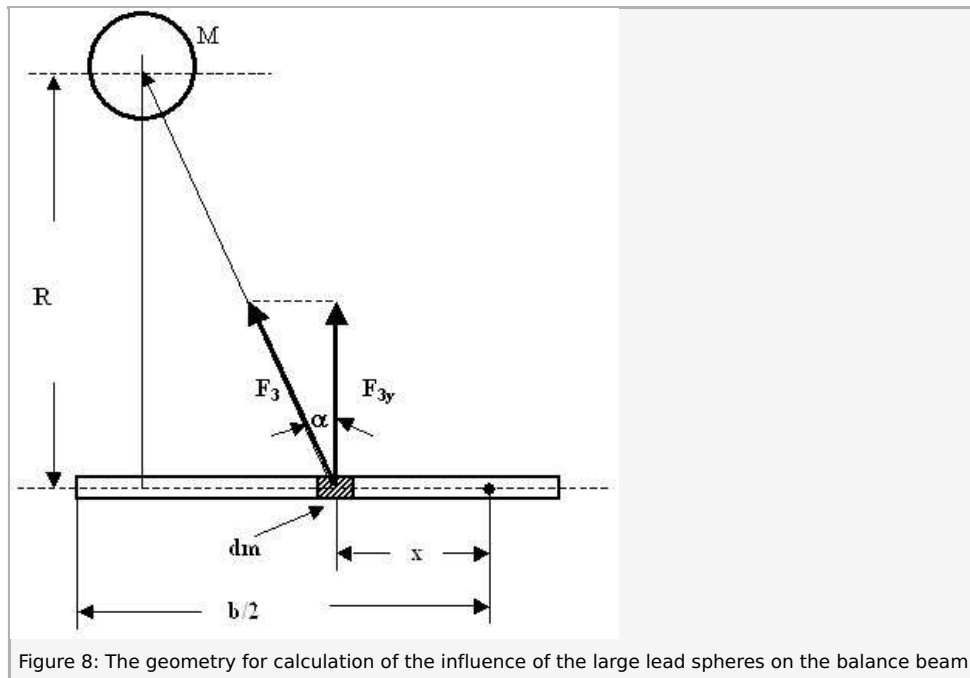


Figure 8: The geometry for calculation of the influence of the large lead spheres on the balance beam

Example data used in the calculations

The values for each individual instrument may differ from the example data. You may wish to determine exact data using a vernier calliper and precision balance.

M	Average mass of a large lead sphere	= 1.049 kg
m	Average mass of a small lead sphere	= $14.50 \cdot 10^{-3}$ kg
r_s	Average radius of a small lead sphere	= $0.67 \cdot 10^{-2}$ m
r_L	Average radius of a large sphere	= $2.82 \cdot 10^{-2}$ m
d	Distance of the axis of rotation to the centre point of a small lead sphere	= $6.66 \cdot 10^{-2}$ m
R	Distance between the centre points of small and large lead sphere in an extreme position	= $4.62 \cdot 10^{-2}$ m
m_b	Mass of the balance beam	= $7.05 \cdot 10^{-3}$ kg
l_b	Length of the balance beam	= $14.9 \cdot 10^{-2}$ m
w_b	Width of the balance beam	= $1.29 \cdot 10^{-2}$ m
d_b	Thickness of the balance beam	= $0.14 \cdot 10^{-2}$ m
W	Distance between the outer surfaces of the glass plates	= $3.5 \cdot 10^{-2}$ m
r_h	Radius of the beam hole for the sphere holding	= $0.45 \cdot 10^{-2}$ m
ρ_{Al}	Density of the aluminium beam	= $2.7 \cdot 10^3$ kg m ⁻³
m_h	Mass of the holes in the balance beam	= $0.34 \cdot 10^{-3}$ kg
f_1	Correction factor	= 0.0352
f_2	Correction factor	= 0.2